Dynamics of Analog Decoders for Different Message Representation Domains

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Introduction

- Different domains can be used for representing messages in iterative decoding.

- In probability domain, we directly use the probability that the received bit is zero (i.e., p).

- Other popular domains are:

$$\delta = 2p - 1 \qquad \text{(likelihood difference)}$$

$$LR = \frac{p}{1-p} \qquad \text{(likelihood ratio)}$$

$$LLR = \log\left(\frac{p}{1-p}\right) \qquad \text{(log-likelihood ratio)}$$

- In ideal iterative decoding (ideal conventional digital discrete-time iterative decoder), message representation domain has no effect on decoding performance.

- However, when accuracy is limited (due to clipping, quantization error, …) different decoding performances can be observed for different domains.

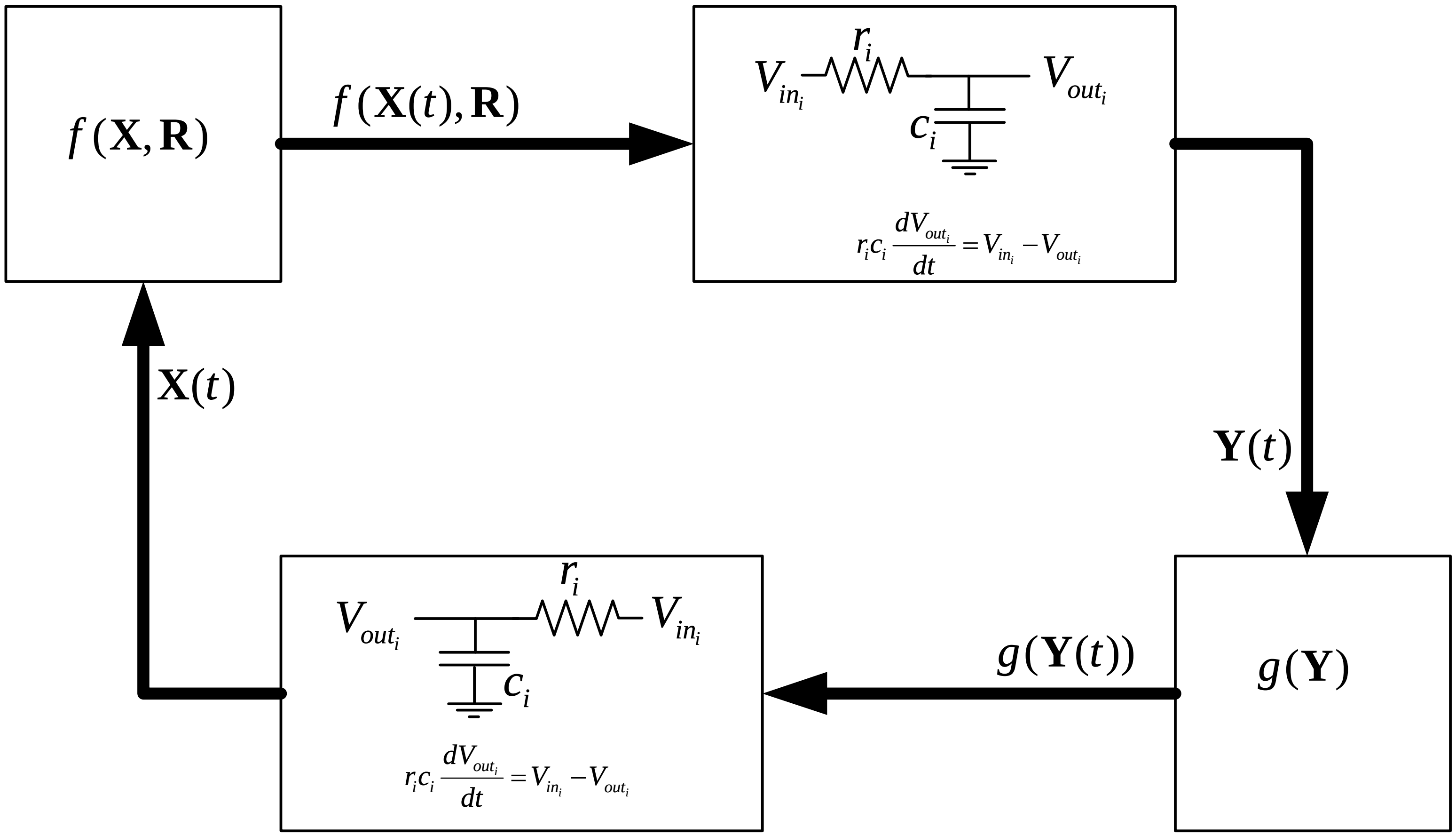
Introduction

- In an abstract level, an ideal analog decoder has the same processing nodes as its digital iterative decoder counterpart, but it necessitates continuity of the messages that are passed through the edges.

- Different analog decoding groups have implemented analog decoders in various domains. For example, Munich decoder used *LLR* and ETH decoder used p (or δ), while both had a similar processing core.

- "Would it change anything if we switch message representation domain in an ideal analog decoder?" We are trying to answer this question.

Introduction



A simple model for analog iterative decoding; here, f and g denote operations in variable nodes and check nodes, respectively.
X and **Y** are message vectors (of size ξ) that are passed between nodes.

Introduction

We define **Z** as:

$$\mathbf{Z}(t) = \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{bmatrix}$$

and *h* function as:

$$h(\mathbf{Z}(t)) = \begin{bmatrix} g\left([\mathbf{0}_{l\times l} \quad \mathbf{I}_{l\times l}] \times \mathbf{Z}(t)\right) \\ f\left([\mathbf{I}_{l\times l} \quad \mathbf{0}_{l\times l}] \times \mathbf{Z}(t), \mathbf{R}\right) \end{bmatrix}$$

It can be shown that in a conventional discrete-time iterative decoder the next state (l+1) is generated based on the current state (l), as follows:

$$\mathbf{Z}_{\omega}^{(l+1)} = h_{\omega}\left(\mathbf{Z}_{\omega}^{(l)}, \mathbf{R}_{\omega}\right)$$

where ω denotes message representation domain and l ($l \geq 0$) denotes the iteration number and $\mathbf{Z}_{\omega}^{(0)}$ is defined properly to initialize the decoder. For analog decoders, dynamic equation is given by:

$$\dot{\mathbf{Z}}_{\omega} = h_{\omega}\left(\mathbf{Z}_{\omega}, \mathbf{R}_{\omega}\right) - \mathbf{Z}_{\omega}$$

where all time constants are assumed equal to 1.

Analog Decoding in *LLR* Domain

Dynamic equation for an *LLR*-based analog decoder is given by:

$$\dot{\mathbf{Z}}_{LLR} = h_{LLR}\left(\mathbf{Z}_{LLR}, \mathbf{R}_{LLR}\right) - \mathbf{Z}_{LLR}$$

We consider the *LLR* domain as the reference domain. In the next slides, we translate dynamic equations in other domains to this domain to facilitate the comparison.

Analog Decoding in *LR* Domain

Let z_i be the *i*-th element of $\mathbf{Z}_{LLR}$ and *EXP* be the element-wise exponential function, then the dynamic equation in *LR* domain is given by:

$$\dot{\mathbf{Z}}_{LR} = h_{LR}(\mathbf{Z}_{LR}, \mathbf{R}_{LR}) - \mathbf{Z}_{LR} = \exp\left(\begin{bmatrix} z_1 & .. & 0 & .. & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & .. & z_i & .. & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & .. & 0 & .. & z_{2\xi} \end{bmatrix}\right)\dot{\mathbf{Z}}_{LLR}$$

$$= EXP\left(h_{LLR}(\mathbf{Z}_{LLR}, \mathbf{R}_{LLR})\right) - EXP(\mathbf{Z}_{LLR})$$

and, consequently by moving from *LR* to *LLR* domain, we have:

$$\dot{\mathbf{Z}}_{LLR} = EXP\left(h_{LLR}(\mathbf{Z}_{LLR}, \mathbf{R}_{LLR}) - \mathbf{Z}_{LLR}\right) - \mathbf{1}$$

where **1** is a column vector of ones.

Analog Decoding in *LR* Domain

It is known that for any non-zero x:

$$x < \exp(x) - 1$$

So, by recalling dynamic equations derived in *LLR* and *LR* domains:

$$LLR: \quad \dot{\mathbf{Z}}_{LLR} = h_{LLR}(\mathbf{Z}_{LLR}, \mathbf{R}_{LLR}) - \mathbf{Z}_{LLR}$$

$$LR: \quad \dot{\mathbf{Z}}_{LLR} = EXP(h_{LLR}(\mathbf{Z}_{LLR}, \mathbf{R}_{LLR}) - \mathbf{Z}_{LLR}) - \mathbf{1}$$

We conclude that *LLR*-based and *LR*-based analog decoders follow different trajectories.

It is easy to verify that if one of these two types of decoders converges to a fixed point and $(\dot{\mathbf{Z}}_{LLR} = \mathbf{0})$ then that point will be a fixed point for the other analog decoder too. However, nothing is guaranteed for the convergence of the second analog decoder to the fixed point.

Analog Decoding in p Domain

Let *TANH*, *COSH*, and *SINH* denote element-wise *tanh*, *cosh*, and *sinh*, respectively. The dynamic equation in p domain is then given by:

$$\dot{\mathbf{Z}}_p = h_p(\mathbf{Z}_p, \mathbf{R}_p) - \mathbf{Z}_p = \frac{1}{4}\left(\mathbf{1} - TANH^2(\mathbf{Z}_{LLR}/2)\right)\dot{\mathbf{Z}}_{LLR}$$

$$= \frac{1}{2}\left(\mathbf{1} + TANH\left(h_{LLR}(\mathbf{Z}_{LLR}, \mathbf{R}_{LLR})/2\right)\right) - \frac{1}{2}\left(\mathbf{1} + TANH(\mathbf{Z}_{LLR}/2)\right)$$

and thus

$$\dot{\mathbf{Z}}_{LLR} = 2\left(TANH\left(h_{LLR}(\mathbf{Z}_{LLR}, \mathbf{R}_{LLR})/2\right)COSH^2(\mathbf{Z}_{LLR}/2)\right) - SINH(\mathbf{Z}_{LLR})$$

Since *tanh* is bounded between -1 and 1, it can be verified that element-wise the following bounds exist:

$$-\mathbf{1} - EXP(\mathbf{Z}_{LLR}) \leq \dot{\mathbf{Z}}_{LLR} \leq \mathbf{1} + EXP(-\mathbf{Z}_{LLR})$$

These bounds differentiate a p-based analog decoder from the *LLR*-based and *LR*-based analog decoders, though again fixed points are identical.

Analog Decoding in δ Domain

Dynamic equation in δ domain is given by:

$$\dot{\mathbf{Z}}_{\delta} = h_{\delta}(\mathbf{Z}_{\delta}, \mathbf{R}_{\delta}) - \mathbf{Z}_{\delta} = \frac{1}{2}\left(\mathbf{1} - TANH^{2}(\mathbf{Z}_{LLR}/2)\right)\dot{\mathbf{Z}}_{LLR}$$

$$= TANH\left(h_{LLR}(\mathbf{Z}_{LLR}, \mathbf{R}_{LLR})/2\right) - TANH(\mathbf{Z}_{LLR}/2)$$

and thus

$$\dot{\mathbf{Z}}_{LLR} = 2\left(TANH\left(h_{LLR}(\mathbf{Z}_{LLR}, \mathbf{R}_{LLR})/2\right)COSH^{2}(\mathbf{Z}_{LLR}/2)\right) - SINH(\mathbf{Z}_{LLR})$$

This equation is same as the dynamic equations for ***p***-based analog decoders and thus we conclude that these two types of decoders are identical.

Conclusions

- Dynamic equations of analog decoders in *LLR*, *LR*, *p*, and δ domains were compared. It was shown that while all have identical fixed-points, only *p*-based and δ-based analog decoders are identical.

- Deriving *h* function in general is not easy, however, we have already shown that an analog min-sum (MS) decoder is in fact a piece-wise linear system in *LLR* domain. We plan to apply these new results to MS decoders.

- This study does not provide any information about the statistical performance of analog decoders. It only indicates dynamic equations are not identical and different trajectories are followed.

- Simulation results, which are based on applying forward Euler (FE) method to corresponding dynamic equations, show that using different message representation domains causes different error decoding performances for analog decoders [14].